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Hamiltonian systems on quantized spaces

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Abstract. In this paper we first characterize the Lie algebra of derivations of the three-dimensional Manin quantum space as the semi-direct product of the Lie algebra of its inner derivations and the threefold generalized Virasoro algebra with central charge zero. Then we consider Hamiltonian systems on the quantum plane and we prove that the set of Hamiltonian derivations is a Virasoro algebra with central charge zero. Moreover, we show that the only possible motions on the quantum plane come from quadratic Hamiltonians and we find the solutions of the corresponding Hamilton equations explicitly.

0. Introduction

Classical and quantum mechanics on q-deformed spaces have been studied by some authors. Most of these works are concerned with Hamiltonian systems [1–3], but there are also some works concerning the Lagrangian formalism on the quantum plane [2, 3]. To construct the classical mechanics on the quantum plane one can use the q-deformed symplectic structure obtained by the q-deformation of the natural symplectic structure of the plane and obtain the equations of motion in the form $\mathrm{d}x/\mathrm{d}t = \{H, x\}_q$, $\mathrm{d}p/\mathrm{d}t = \{H, p\}_q$ [1, 2]. Unfortunately, the q-deformed Poisson bracket has nothing in common with the usual Poisson bracket, unless it is bilinear, and its only use is in writing the equations of motion as above. However, most of the very interesting facts of classical mechanics are absent here. It is unfortunate that, in general, it is not true that $\{H, H\}_q = 0$, and $\{H, f\}_q = 0$ does not imply $\{H^2, f\}_q = 0$.

It is well known that there are two approaches to classical mechanics based on the symplectic structure of the phase space [4, 5]. The first is the state approach and the second is the observable approach. In these approaches the coordinate and the momentum functions appear like other observables and they all satisfy the same equation. A new interpretation of the quantum spaces is given in [6]. According to this interpretation the two approaches to classical mechanics are also suitable for the case of quantum spaces. To be more precise, let $\mathcal{M}_q$—for the notation and conventions see the following pages—denote the A-algebra of Q-meromorphic functions and let π be the canonical q-deformed Poisson structure on $\mathcal{M}_q$. By a Hamiltonian system we mean a triple $(\mathcal{M}_q, \pi, z)$, where z is a π-Hamiltonian element of $\mathcal{M}_q$. Here also, just like the ordinary case, in the observable approach by the

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motion of the above system we mean a strongly differentiable one-parameter local group of automorphisms of the system ϕ_t, satisfying the condition

$$\forall f \in \mathcal{M}_q \qquad \frac{\mathrm{d}\phi_t(f)}{\mathrm{d}t} = \{z, \phi_t(f)\} \qquad \phi_0(f) = f$$

and in the state approach by the motion of the system with initial values (x, p) we mean a path $t \to (x(t)\ p(t))$ in R^2 such that for each $f \in \mathcal{M}_q$

$$\frac{\mathrm{d}}{\mathrm{d}t} f(x(t)\ p(t)) = \{z, f\}(x(t)\ p(t)) \qquad x(0) = x \qquad p(0) = p.$$

In this paper following [1, 2] we follow the observable approach, and accept that the mass m of a particle moving on a straight line, like its coordinate and momentum, is an operator. Therefore, we have the following commutation relations

$$mx = qxm \qquad mp = qpm.$$

See also [1, 2]. We mention that, according to the state approach on a quantum plane, the mass m is a real number. So, the two above-mentioned approaches to classical mechanics on quantum spaces are not equivalent. As we will see, we can not consider an arbitrary element of $\mathcal{M}_q$ as a Hamiltonian. Indeed, to each Hamiltonian element on $\mathcal{M}_q$ there corresponds a unique Hamiltonian derivation and the set of all these derivations is a Virasoro algebra with central charge zero $\mathcal{V}$. However, as we will see for a general Hamiltonian, in our sense the corresponding Hamilton equations do not define any motion in general. When we restrict ourselves to the Hamiltonian systems $(\mathcal{M}_q, \pi, z)$, with z in the subalgebra $sl(2, A)$ of $\mathcal{V}$, and only in this case the Hamilton equations only give the motion of the system and then one can easily see that the possible motions on the quantum spaces are those coming from quadratic Hamiltonians. Clearly these motions are of very restricted types. So, we should look for other quantum manifolds to have motions of other types.

This paper consists of three sections. The structure of the Lie algebra of derivations of the three-dimensional Manin quantum space $\mathcal{M}_q$ is considered in section 1. In this section we show that the Lie algebra of derivations of $\mathcal{M}_q$ is the semi-direct product of the Lie algebra of inner derivations of this algebra and the threefold generalized Virasoro algebra with central charge zero, i.e. a Lie algebra generated by the family $\{L_i^m | m = 1, 2, 3 \text{ and } i \in Z\}$ and the multiplication rule

$$[L_i^m, L_j^n] = jL_{i+j}^n - iL_{i+j}^m.$$

Section 2 is devoted to Hamiltonian systems on $\mathcal{M}_q$. In this section we define the notion of a Hamiltonian derivation. We then prove that the set of all such derivations is a Virasoro algebra with central charge zero. Finally, the Hamiltonian equations on $(\mathcal{A}\ \{,\}_q)$ are solved in section 3.

Before going any further we remind that in this paper by A we mean the C-algebra of all absolutely convergent power series $\sum_{i>-\infty} c_i q^i$ on $]0, 1]$ with values in C. Consider the following commutation relations between x, p, m,

$$x^a x^b = x^{a+b} \qquad p^a p^b = p^{a+b} \qquad m^a m^b = m^{a+b}$$
$$p^a x^b = q^{ab} x^b p^a \qquad m^a x^b = q^{ab} x^b m^a \qquad m^a p^b = q^{ab} p^b m^a$$

where a and b are in Z. By $\mathcal{M}_q$ we mean the A-algebra of Q-meromorphic functions of the form

$$\sum_{i,j,k \gg -\infty} a_{ijk}(q) x^i p^j m^k$$

where the sign '≫' under the 'Σ' means that the indices i, j, k are bounded below. By using the above commutation relations one can easily see that $\mathcal{M}_q$ is closed under the Cauchy product. Therefore, it is an A-algebra. The value of an element $z \in \mathcal{M}_q$ written in the above form at a point $(r\, s\, t)$ in R^3 is

$$\sum_{i,j,k \gg -\infty} a_{ijk}(q) r^i s^j t^k$$

which is absolutely convergent by definition. The concept of a 'Q-meromorphic function' is a generalization of the concept of a 'Q-analytic function' given in [4].

The subalgebra of $\mathcal{M}_q$ consisting of all

$$z = \sum_{i,j \geqslant 0, k \gg -\infty} a_{ijk}(q) x^i p^j m^k$$

and the subalgebra consisting of all

$$z = \sum_{k \gg -\infty} c_i(q) m^k$$

will be denoted by $\mathcal{A}$ and M, respectively. We also remind that all the above-mentioned function spaces are endowed with the natural locally convex structures. Finally, throughout the paper the sign '−' on a 'Σ' means that the 'Σ' has finite support.

1. Derivations of $\mathcal{M}_q$

Let $D : \mathcal{M}_q \to \mathcal{M}_q$ be a derivation. Assume that

$$D(x) = \sum_{i,j,k \gg -\infty} a_{ijk}(q) x^{i+1} p^j m^k \qquad D(p) = \sum_{i,j,k \gg -\infty} b_{ijk}(q) x^i p^{(j+1)} m^k$$

and

$$D(m) = \sum_{i,j,k \gg -\infty} c_{ijk}(q) x^i p^j m^{k+1}.$$

Then, direct calculation shows that

$$(q^k - q^i) a_{ijk} = (q^{k+j} - 1) b_{ijk} \qquad (q^{i+j} - 1) b_{ijk} = (q^i - q^k) c_{ijk}$$
$$(q^{j+k} - 1) c_{ijk} = (1 - q^{i+j}) a_{ijk}.$$

Therefore, for $q \neq 0$,

$$k \neq 0 \quad \text{and} \quad a_{ijk} \neq 0 \Leftrightarrow k + j \neq 0 \quad \text{and} \quad b_{ijk} \neq 0 \Leftrightarrow i + j \neq 0 \quad \text{and} \quad c_{ijk} \neq 0. \tag{1}$$

The derivation D is called type 1 if, for $i = -j = k$, $a_{ijk} = b_{ijk} = c_{ijk} = 0$. The set of all derivations of type 1 will be denoted by $\mathcal{D}_1$.

Let D be a derivation of type 1 and let $D(x)$, $D(p)$ and $D(m)$ be as above. Let $q \neq 1$ and

$$z = \sum_{ijk \gg -\infty} (q^{k+j} - 1)^{-1} a_{ijk} x^i p^j m^k.$$

Clearly z is a well defined element of $\mathcal{M}_q$ and we have

$$[z, x] = \sum_{i,j,k \gg -\infty} a_{ijk} x^{i+1} p^j m^k = D(x)$$

$$[z, p] = \sum_{i,j,k \gg -\infty} (q^{j+k} - 1)^{-1} a_{ijk} (q^k - q^i) x^i p^{j+1} m^k = \sum_{i,j,k \gg -\infty} b_{ijk} x^i p^{j+1} m^k = D(p)$$

$$[z, m] = \sum_{i,j,k \gg -\infty} (q^{k+j} - 1)^{-1} a_{ijk} (1 - q^{i+j}) x^i p^j m^{k+1} = \sum_{i,j,k \gg -\infty} c_{ijk} x^i p^j m^{k+1} = D(m).$$

Therefore, for each $y \in \mathcal{M}_q$, $D(y) = [z, y]$. Since elements of $\mathcal{M}_q$ of the form $z = \overline{\sum_i} a_i x^i p^{-i} m^i$ are in the centre of $\mathcal{M}_q$, *each derivation of type 1 is an inner derivation and vice versa*.

Let $D : \mathcal{M}_q \to \mathcal{M}_q$ be a derivation. Assume that

$$D(x) = \overline{\sum_{i \gg -\infty}} a_i x^{i+1} p^{-1} m^i \qquad D(p) = \overline{\sum_{i \gg -\infty}} b_i x^i p^{1-i} m^i$$

$$D(m) = \overline{\sum_{i \gg -\infty}} c_i x^i p^{-i} m^{i+1}.$$

Then D is called a derivation of type 2. Direct calculation shows that in this case we have

$$D(x)x = xD(x) \qquad qD(x)p = pD(x) \qquad qD(x)m = mD(x)$$
$$qxD(p) = D(p)x \qquad D(p)p = pD(p)$$
$$qD(p)m = mD(p) \qquad qxD(m) = D(m)x$$
$$qpD(m) = D(m)p \qquad D(m)m = mD(m).$$

Moreover, as one can see easily, *a derivation* $D : \mathcal{M}_q \to \mathcal{M}_q$ *is of type 2 if and only if it satisfies one of the above nine relations*.

The set of all derivations of type 2 will be denoted by $\mathcal{D}_2$.

Let

$$A = \overline{\sum_{i \gg -\infty}} a_i(q) x^{i+1} p^{-i} m^i \qquad B = \overline{\sum_{i \gg -\infty}} b_i(q) x^i p^{1-i} m^i$$

and

$$C = \overline{\sum_{i \gg -\infty}} c_i(q) x^i p^{-i} m^{i+1}.$$

Then the linear operators

$$D_1, D_2, D_3 : \mathcal{M}_q \to \mathcal{M}_q$$

given by

$$D_1(x^a p^b m^c) = \sum_{i+j=a-1} x^i A x^j p^b m^c \qquad D_2(p) = \sum_{i+j=b-1} x^a p^i B p^j m^c$$

$$D_3(m) = \sum_{i+j=c-1} x^a p^b m^i C m^j$$

and

$$D_1(p) = D_1(m) = 0 \qquad D_2(x) = D_2(m) = 0 \qquad D_3(x) = D_3(p) = 0$$

are derivations of type 2.

We call D_1 an x-derivation, D_2 a p-derivation and D_3 an m-derivation. It is clear that each derivation of type 2, $D : \mathcal{M}_q \to \mathcal{M}_q$, can be written uniquely as the sum of an x-derivation D_1, a p-derivation D_2 and an m-derivation D_3, and

$$D_1(x) = D(x) \qquad D_2(p) = D(p) \qquad D_3(m) = D(m).$$

We mention in passing that if $D : \mathcal{M}_q \to \mathcal{M}_q$ is a derivation of type 2, then for each monomial $z = x^a p^b m^c$ we have $zD(z) = D(z)z$.

Let

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial p}, \frac{\partial}{\partial m} : \mathcal{M}_q \to \mathcal{M}_q$$

be linear operators given by

$$\frac{\partial}{\partial x}(x^i p^j m^k) = i x^{i-1} p^j m^k \qquad \frac{\partial}{\partial p}(x^i p^j m^k) = q^{-i} j x^i p^{j-1} m^k$$

and

$$\frac{\partial}{\partial m}(x^i p^j m^k) = q^{-(i+j)} k x^i p^j m^{k-1}.$$

Assume that $D : \mathcal{M}_q \to \mathcal{M}_q$ is a derivation. Then D can be written uniquely as

$$D = D(x)\frac{\partial}{\partial x} + D(p)\frac{\partial}{\partial p} + D(m)\frac{\partial}{\partial m}.$$

Because, for a, b and c in $\mathbb{N}$, we have

$$\begin{aligned} D(x^a p^b m^c) &= x^a D(p^b) m^c + D(x^a) p^b m^c + x^a p^b D(m^c) \\ &= b x^a p^{b-1} D(p) m^c + a x^{a-1} D(x) p^b m^c + c x^a p^b m^{c-1} D(m) \\ &= q^{-a}[b D(p) x^a p^{b-1} m^c] + a x^{b-1} D(x) p^b m^c + q^{-(i+j)} c D(m) x^a p^b m^{c-1} \\ &= \left[D(p)\frac{\partial}{\partial p} + D(x)\frac{\partial}{\partial x} + D(m)\frac{\partial}{\partial m} \right](x^a p^b m^c). \end{aligned}$$

If $D : \mathcal{M}_q \to \mathcal{M}_q$ is a derivation of type 2, then

$$D_1 = D(x)\frac{\partial}{\partial x} \qquad D_2 = D(p)\frac{\partial}{\partial p} \qquad D(m)\frac{\partial}{\partial m}$$

are derivations. To prove this, it is sufficient to prove that $D(x)\partial/\partial x$, $D(p)\partial/\partial p$ and $D(m)\partial/\partial m$ are derivations. Let a, b, c, r, s and t be in $\mathbb{N}$. Then

$$\begin{aligned} D(x)\frac{\partial}{\partial x}(x^a p^b m^c x^r p^s m^t) &= q^{r(b+c)+sc} D(x)\frac{\partial}{\partial x}(x^{a+r} p^{b+s} m^{c+t}) \\ &= q^{r(b+c)+sc}(a+r) D(x) x^{a+r-1} p^{b+s} m^{c+t} \\ &= x^a p^b m^c D(x)\frac{\partial}{\partial x}(x^r p^s m^t) + D(x)\frac{\partial}{\partial x}(x^a p^b m^c) x^r p^s m^t. \end{aligned}$$

In the same way we see that $D(p)\partial/\partial p$ and $D(m)\partial/\partial m$ are derivations.

The set of all derivations of $\mathcal{M}_q$ which is clearly an A-Lie algebra will be denoted by $\mathcal{D}$.

Now it is clear that $\mathcal{D}_2$ *with*

$$[D_1, D_2] = D_1 D_2 - D_2 D_1$$

is an A-Lie algebra. Moreover, *each derivation of $\mathcal{M}_q$ can be written uniquely as $D = D_1 + D_2$, where $D_1 \in \mathcal{D}_1$ and $D_2 \in \mathcal{D}_2$*. Furthermore, *$\mathcal{D}_2$ is an ideal of $\mathcal{D}$*. Therefore, *$\mathcal{D}$ is the semi-direct product of $\mathcal{D}_1$ and $\mathcal{D}_2$*.

Let $X^i = (q^{i/2}x)^i$, $P^j = (q^{j/2}p)^j$, $M^k = (q^{k/2}m)^k$ and

$$L_i^1 = X^i P^{-i} M^i \frac{\partial}{\partial x} \qquad L_{-j}^2 = X^j P^{-j} M^j \frac{\partial}{\partial p} \qquad \text{and} \qquad L_k^3 = X^k P^{-k} M^k \frac{\partial}{\partial m}.$$

Then clearly we have

$$[L_i^l, L_j^n] = jL_{(i+j)}^n - iL_{(i+j)}^l.$$

Therefore, $\mathcal{D}_2$ *is a threefold Virasoro algebra with central charge zero*.

2. Hamiltonian systems on the quantum plane

In this section we endow $\mathcal{M}_q$ with the canonical q-deformed Poisson structure

$$\pi = q^{-1/2}\frac{\partial}{\partial x} \otimes \frac{\partial}{\partial p} - q^{1/2}\frac{\partial}{\partial p} \otimes \frac{\partial}{\partial x}.$$

The associated q-deformed Poisson bracket will be denoted by $\{,\}_q$. More precisely, for each two elements $f, g \in \mathcal{M}_q$ we have

$$\{f, g\}_q = q^{-1/2}\frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - q^{1/2}\frac{\partial f}{\partial p}\frac{\partial g}{\partial x}.$$

An element $z \in \mathcal{M}_q$ is called Hamiltonian if the mapping

$$X_z : \mathcal{M}_q \to \mathcal{M}_q$$

defined by

$$X_z(f) = \{z, f\}_q$$

is a derivation. In this case X_z is called a Hamiltonian derivation.

Assume that for $z \in \mathcal{M}_q$, X_z is a derivation. Since $X_z(m) = 0$ it is necessarily a derivation of type 2. Let X_z be a derivation. Suppose that

$$z = \sum_{i,j,k \gg -\infty} a_{ijk}(q) x^i p^j m^k.$$

Then

$$X_z(x) = -\sum_{i,j,k} j a_{ijk}(q) q^{-i+1/2} x^i p^{j-1} m^k \qquad X_z(p) = \sum_{i,j,k} i a_{ijk}(q) q^{-1/2} x^{i-1} p^j m^k$$

and $X_z(m) = 0$. However, since X_z is a derivation of type 2 as we have seen, $i = k+1$ and $j = 1-k$. Conversely, assume that

$$z = \overline{\sum_{k \gg -\infty}} a_k(q) x^{k+1} p^{1-k} m^k.$$

Then

$$A = X_z(x) = -\overline{\sum_{k \gg -\infty}} a_k(q) q^{-(1+2k)/2} (1-k) x^{1+k} p^{-k} m^k$$

$$B = X_z(p) = \overline{\sum_{k \gg -\infty}} a_k(q) q^{-1/2} (1+k) x^k p^{1-k} m^k$$

and $C = X_z(m) = 0$. Hence X_z is a derivation of type 2. Therefore, $z \in \mathcal{M}_q$, X_z is Hamiltonian if and only if it is of the above form.

The set of all Hamiltonian elements of $\mathcal{M}_q$ will be denoted by $H(\mathcal{M}_q)$. It is clear that $H(\mathcal{M}_q)$ is an A-module. Let $z^1 = x^{k+1}p^{1-k}m^k$ and $z^2 = x^{l+1}p^{1-l}m^l$. Direct computation shows that

$$\{z^1, z^2\}_q = 2(k-l)q^{-(1/2+kl)}x^{1+(k+l)}p^{1-(k+l)}m^{k+l}.$$

Therefore, for each two elements z^1, z^2 in $H(\mathcal{M}_q)$, $\{z^1, z^2\}_q \in H(\mathcal{M}_q)$. Moreover, $\{z^1, z^2\}_q = -\{z^2, z^1\}_q$. Now let z^1 and z^2 be as above and $z^3 = x^{n+1}p^{1-n}m^n$. Then

$$\{\{z^1, z^2\}_q, z^3\}_q + \{\{z^2, z^3\}_q, z^1\}_q + \{\{z^3, z^1\}_q, z^2\}_q = 0.$$

Therefore $(H(\mathcal{M}_q)\{,\}_q)$ is an A-Lie algebra, with centre M. Let z^1 and z^2 be as above. Then

$$X_{\{z^1,z^2\}_q}(x) = 2(k-l)(1-k-l)q^{-(1+k+l+kl)}x^{1+(k+l)}p^{1-(k+l)}m^{k+l} = [X_{z^1}\, X_{z^2}](x)$$

$$X_{\{z^1,z^2\}_q}(p) = 2(k-l)(1+k+l)q^{-(1+kl)}x^{k+l}p^{1-(k+l)}m^{k+l} = [X_{z^1}\, X_{z^2}](p)$$

and

$$X_{\{z^1,z^2\}_q}(m) = 0.$$

From the above considerations we see that the mapping

$$X : H(\mathcal{M}_q) \to \mathcal{D}$$

given by $X(z) = X_z$ is a homomorphism of A-Lie algebras with kernel M.

Let $z_n \in H(\mathcal{M}_q)$ be defined as follows

$$z_n = 1/2q^{(1-n^2)/2}x^{1+n}p^{1-n}m^n \qquad n \in Z.$$

Then $\{z_m, z_n\} = (m-n)z_{m+n}$. Therefore, *the Lie algebra of Hamiltonian derivations of $\mathcal{M}_q$ is the Virasoro algebra with central charge zero*. This algebra will be denoted by $\mathcal{V}$.

Let z be in $H(\mathcal{M}_q)$. Then for each f in $\mathcal{M}_q$ we have

$$\{z, f\}_q = \{z, x\}_q \frac{\partial f}{\partial x} + \{z, p\}_q \frac{\partial f}{\partial p}$$

because

$$\{z, f\}_q = X_z(f) = X_z(x)\frac{\partial f}{\partial x} + X_z(p)\frac{\partial f}{\partial p} = \{z, x\}_q \frac{\partial f}{\partial x} + \{z, p\}_q \frac{\partial f}{\partial p}.$$

By a Hamiltonian system on the quantum plane we mean a triple $(\mathcal{M}_q, \pi, z)$, where π is the canonical q-deformed Poisson structure on $\mathcal{M}_q$ and $z \in H(\mathcal{M}_q)$. Let ϕ_t be a strongly differentiable one-parameter local group of automorphisms of the q-deformed Poisson structure $(\mathcal{M}_q, \pi)$. We say that ϕ_t defines the motion of the system $(\mathcal{M}_q, \pi, z)$, if for each $f \in \mathcal{M}_q$

$$\frac{\mathrm{d}f_t}{\mathrm{d}t} = \{z_t, f_t\}_q \qquad f_0 = f$$

where for each $f \in \mathcal{M}_q$, $\phi_t(f)$ is denoted by f_t.

Proposition. A necessary and sufficient condition for ϕ_t to define the motion of the Hamiltonian system $(\mathcal{M}_q, \pi, z)$ is that, for each t, $z_t \in H(\mathcal{M}_q)$ and x_t, p_t and m_t satisfy the following equations:

$$\frac{\mathrm{d}x_t}{\mathrm{d}t} = \{z_t, x_t\}_q \qquad \frac{\mathrm{d}p_t}{\mathrm{d}t} = \{z_t, p_t\}_q \qquad \frac{\mathrm{d}m_t}{\mathrm{d}t} = \{z_t, m_t\}_q.$$

Proof. The condition is clearly necessary. To prove that the condition is sufficient, assume that x_t, p_t and m_t satisfy the above equations and $z_t \in H(\mathcal{M}_q)$. Then

$$x_t \frac{\mathrm{d}x_t}{\mathrm{d}t} = x_t\{z_t, x_t\}_q \qquad p_t \frac{\mathrm{d}p_t}{\mathrm{d}t} = p_t\{z_t, p_t\}_q \qquad m_t \frac{\mathrm{d}m_t}{\mathrm{d}t} = m_t\{z_t, m_t\}_q.$$

As we have seen earlier, for each t, X_{z_t} is a derivation. Therefore,

$$x_t\{z_t, x_t\}_q = x_t X_{z_t}(x_t) = X_{z_t}(x_t)x_t = \frac{\mathrm{d}x_t}{\mathrm{d}t}x_t$$
$$p_t\{z_t, p_t\}_q = p_t X_{z_t}(p_t) = X_{z_t}(p_t)p_t = \frac{\mathrm{d}p_t}{\mathrm{d}t}p_t$$
$$m_t\{z_t, m_t\}_q = m_t X_{z_t}(m_t) = X_{z_t}(m_t)m_t = \frac{\mathrm{d}m_t}{\mathrm{d}t}m_t.$$

Now let $f = x^i p^j m^k$. Then $f_t = x_t^i p_t^j m_t^k$ and

$$\begin{aligned}\frac{\mathrm{d}f_t}{\mathrm{d}t} &= i x_t^{i-1}\frac{\mathrm{d}x_t}{\mathrm{d}t}p_t^j m_t^k + x_t^i j p_t^{j-1}\frac{\mathrm{d}p_t}{\mathrm{d}t}m_t^k + x_t^i p_t^j k m_t^{k-1}\frac{\mathrm{d}m_t}{\mathrm{d}t}\\ &= \frac{\mathrm{d}x_t}{\mathrm{d}t}(i x_t^{i-1}p_t^j m^k) + \frac{\mathrm{d}p_t}{\mathrm{d}t}(q^{-i} j x_t^i p_t^{j-1} m^k) + \frac{\mathrm{d}m_t}{\mathrm{d}t}(q^{-(i+j)} k x_t^i p_t^j m_t^{k-1})\\ &= X_{z_t}(x_t)\frac{\partial f_t}{\partial x_t} + X_{z_t}(p_t)\frac{\partial f_t}{\partial p_t} + X_{z_t}(m_t)\frac{\partial f_t}{\partial x_t} = X_{z_t}(f_t) = \{z_t, f_t\}_q.\end{aligned}$$

Therefore, for each $f \in \mathcal{M}_q$ we have $\mathrm{d}f_t/\mathrm{d}t = \{z_t, f_t\}_q$.

Now since, for each t, $\{z_t, z_t\}_q = 0$, therefore $\mathrm{d}z_t/\mathrm{d}t = 0$. This means that z is an invariant of motion. It is easy to see that any analytic function of z is also an invariant of motion. □

Note that the Hamilton equations on $\mathcal{M}_q$, in general, does not define a motion of the corresponding Hamiltonian system.

3. Hamiltonian systems on $\mathcal{A}_q$

Clearly the subalgebra of $\mathcal{V}$ generated by

$$z_0 = 1/2q^{1/2}xp \qquad z_1 = 1/2x^2m \qquad z_{-1} = 1/2p^2m^{-1}$$

is $sl(2, A)$. Now consider the Hamiltonian system $(\mathcal{A}_q, \pi, z)$, where $z = \alpha z_{-1} + \beta z_1 + \gamma z_0$, and α, β and $\gamma \in A$. The corresponding Hamilton equations are

$$\frac{\mathrm{d}x_t}{\mathrm{d}t} = -q^{-1/2}(\alpha p_t m^{-1} + q^{-1}\gamma x_t) \qquad \frac{\mathrm{d}p_t}{\mathrm{d}t} = q^{-1/2}(\beta x_t m + \gamma p_t)$$

or in matrix form

$$\left(\frac{\mathrm{d}x_t}{\mathrm{d}t} \quad \frac{\mathrm{d}p_t}{\mathrm{d}t}\right) = (x_t\ p_t)\begin{pmatrix} -q^{-1/2}\gamma & q^{-1/2}\beta m \\ -q^{1/2}\alpha m^{-1} & q^{-1/2}\gamma\end{pmatrix}.$$

By solving this linear differential equation with constant coefficients we obtain

$$x_t = \cosh\theta t x - \theta^{-1}\sinh\theta t(q^{-1/2}\gamma x + q^{1/2}\alpha p m^{-1})$$
$$p_t = \cosh\theta t p + \theta^{-1}\sinh\theta t(q^{-1/2}\beta x m + q^{-1/2}\gamma p)$$

where $\theta = (q^{-1/2}\gamma^2 - \alpha\beta)^{1/2}$, $x = x_0$ and $p = p_0$.

Now let q be a constant complex number and let $z^{1/2}$ denotes the non-principal branch of the second root of z. Then we have the following.

(1) Let $\alpha = 1$ and $\beta = \gamma = 0$. In this case we have

$$x_t = x - q^{1/2} p m^{-1} \qquad p_t = p.$$

(2) Let $\alpha = 1$, $\beta = \omega^2$ and $\gamma = 0$. In this case we have

$$x_t = x \cos \omega t - q^{-1/2} p (\omega m)^{-1} \sin \omega t$$
$$p_t = p \cos \omega t + q^{-1/2} \omega x m \sin \omega t.$$

Note that in these two special cases the slight difference between our results and those in [1] comes from the difference between the definitions of the q-deformed Poisson structures given in [1], and also the difference between Hamiltonians comes from different rules of differentiation.

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References

[1] Ya Arefeva I and Volovich I V 1991 *Phys. Lett.* **268B** 179
[2] Malik J R P 1993 *Phys. Lett.* **316B** 257
[3] Lukin M, Stern A and Yakushin I 1993 *J. Phys. A: Math. Gen.* **26** 5115
[4] Abraham R and Marsden E 1978 *Foundations of Mechanics* (Reading, MA: Addison-Wesley)
[5] Arnold V 1975 *Mathematical Methods of Classical Mechanics* (Moscow: Mir)
[6] Shafei Deh Abad A and Milani V 1994 *J. Math. Phys.* **35** 5074